15.4 + 15.5. Applications of double integrals

Prop Consider a lamina which occupies a region D on the xy-plane with density $\rho(x,y)$.

(1) Its mass is $m = \iint_D \rho(x,y)\, dA$.

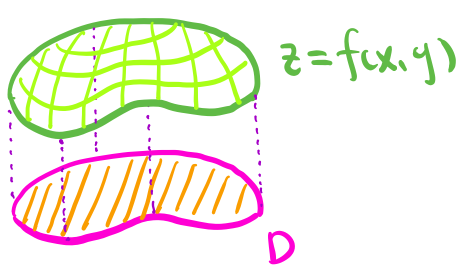
(2) Its center of mass is $(\bar{x}, \bar{y})$ with

$$\begin{cases} \bar{x} = \dfrac{1}{m} \iint_D x\rho(x,y)\, dA \\ \bar{y} = \dfrac{1}{m} \iint_D y\rho(x,y)\, dA \end{cases}$$

Note We will see similar formulas for mass and center of mass with many different types of integrals.

Recall: The length of a graph $y=f(x)$ over an interval $[a,b]$ is $\int_a^b \sqrt{1+f'(x)^2}\, dx$

Prop The area of the graph $z=f(x,y)$ above a domain D is $\iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\, dA$.

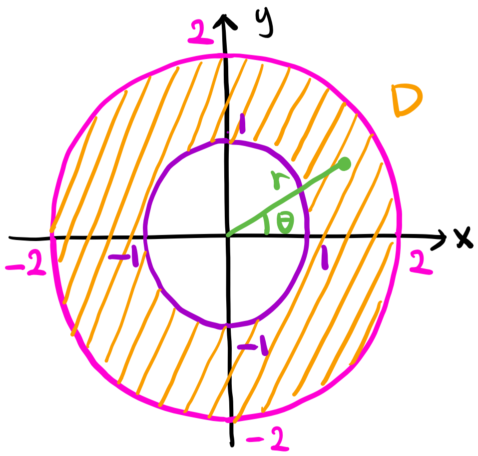


Ex Find the area of the paraboloid $z = x^2+y^2$ which lies between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.

Sol The paraboloid is the graph of $f(x,y) = x^2+y^2$.

$$\Rightarrow \frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y.$$

The domain is $D = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 4\}$.



In polar coordinates, the bounds of D are given by

$0 \le \theta \le 2\pi$ and $1 \le r \le 2$.

$$\text{Area} = \iint_D \sqrt{1+4x^2+4y^2}\, dA$$

Jacobian

$$= \int_0^{2\pi}\int_1^2 \sqrt{1+4r^2} \cdot r\, dr\, d\theta$$

$$(u = 1+4r^2 \Rightarrow du = 8r\,dr)$$

$$= \int_0^{2\pi}\int_5^{17} u^{\frac{1}{2}} \cdot \frac{1}{8}\, du\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} u^{\frac{3}{2}} \Big|_{u=5}^{u=17} d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}(17^{3/2} - 5^{3/2})\, d\theta = \boxed{\frac{\pi}{6}(17^{3/2} - 5^{3/2})}$$

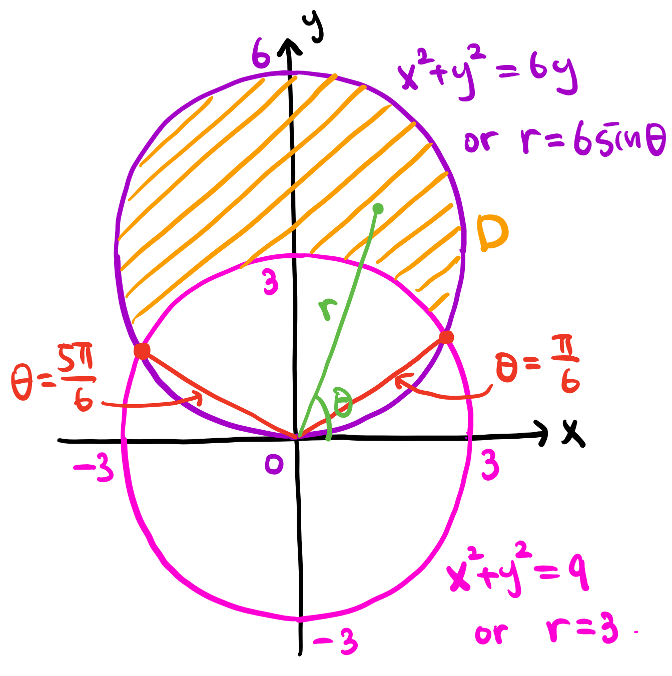
Ex A thin lamina occupies the region D which lies inside the circle $x^2+y^2=6y$ and outside the circle $x^2+y^2=9$. Its density is inversely proportional to the distance from the origin.

(1) Describe D in polar coordinates.

Sol $x^2+y^2=6y \rightsquigarrow x^2+y^2-6y+9=9 \rightsquigarrow x^2+(y-3)^2=9$

$\rightsquigarrow$ a circle of radius 3 and center $(0,3)$.

$x^2+y^2=9 \rightsquigarrow$ a circle of radius 3 and center $(0,0)$.



In polar coordinates:

$x^2+y^2=9 \rightsquigarrow r^2=9 \rightsquigarrow r=3$.

$x^2+y^2=6y \rightsquigarrow r^2=6r\sin\theta$

$\rightsquigarrow r=6\sin\theta$

At the intersections:

$r=3$ and $r=6\sin\theta$

$\Rightarrow 3=6\sin\theta \Rightarrow \sin\theta=\frac{1}{2}$

$\Rightarrow \theta=\frac{\pi}{6}, \frac{5\pi}{6}$.

D is given by $\boxed{\frac{\pi}{6}\le\theta\le\frac{5\pi}{6} \text{ and } 3\le r\le 6\sin\theta}$

(2) Find the center of mass.

<u>Sol</u> The density $\rho(x,y)$ is inversely proportional to the distance from the origin.

$$\Rightarrow \rho(x,y) = \frac{c}{\sqrt{x^2+y^2}} \text{ for some constant } c.$$

Jacobian

$$\text{Mass } m = \iint_D \frac{c}{\sqrt{x^2+y^2}}\,dA = \int_{\pi/6}^{5\pi/6}\int_3^{6\sin\theta} \frac{c}{r}\cdot r\,dr\,d\theta$$

$$= c\int_{\pi/6}^{5\pi/6}\int_3^{6\sin\theta} 1\,dr\,d\theta = c\int_{\pi/6}^{5\pi/6} 6\sin\theta - 3\,d\theta$$

$$= (-6\cos\theta - 3\theta)\Big|_{\theta=\pi/6}^{\theta=5\pi/6} = c(6\sqrt{3}-2\pi).$$

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x,y)\,dA = \frac{1}{m}\iint_D \underbrace{\frac{cx}{\sqrt{x^2+y^2}}}_{\text{odd w.r.t } x}\,dA = 0$$

Symm. about the y-axis

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{1}{m}\iint_D \frac{cy}{\sqrt{x^2+y^2}}\,dA = 0$$

Jacobian

$$= \frac{1}{m}\int_{\pi/6}^{5\pi/6}\int_3^{6\sin\theta} \frac{cr\sin\theta}{r}\cdot r\,dr\,d\theta$$

$$= \frac{c}{m}\int_{\pi/6}^{5\pi/6}\int_3^{6\sin\theta} r\sin\theta\,dr\,d\theta$$

$$= \frac{c}{m}\int_{\pi/6}^{5\pi/6} \frac{r^2}{2}\sin\theta\,\Big|_{r=3}^{r=6\sin\theta}\,d\theta$$

$$= \frac{c}{m}\int_{\pi/6}^{5\pi/6} 18\sin^3\theta - \frac{9}{2}\sin\theta \, d\theta$$

$$= \frac{c}{m}\int_{\pi/6}^{5\pi/6} 18\sin^2\theta \cdot \sin\theta - \frac{9}{2}\sin\theta \, d\theta$$

$$= \frac{c}{m}\int_{\pi/6}^{5\pi/6} 18(1-\cos^2\theta)\sin\theta - \frac{9}{2}\sin\theta \, d\theta$$

$(u = \cos\theta \Rightarrow du = -\sin\theta \, d\theta)$

$$= \frac{c}{m}\int_{\sqrt{3}/2}^{-\sqrt{3}/2} 18(1-u^2)(-1) + \frac{9}{2} \, du$$

$$= \frac{c}{m}\int_{\sqrt{3}/2}^{-\sqrt{3}/2} 18u^2 - \frac{27}{2} \, du$$

$$= \frac{c}{m}\left(6u^3 - \frac{27}{2}\right)\Bigg|_{u=\sqrt{3}/2}^{u=-\sqrt{3}/2}$$

$$= \frac{c}{c(6\sqrt{3}-2\pi)} \cdot 9\sqrt{3} = \frac{9\sqrt{3}}{6\sqrt{3}-2\pi}$$

$\Rightarrow$ The center of mass is $\boxed{\left(0, \frac{9\sqrt{3}}{6\sqrt{3}-2\pi}\right)}$